

Algebraically general, gravito-electric rotating dust

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(Dated: June 17, 2008)

The class of gravito-electric, algebraically general, rotating ‘silent’ dust space-times is studied. The main invariant properties are deduced. The number t_0 of functionally independent zero-order Riemann invariants satisfies $1 \leq t_0 \leq 2$. The solutions may be subdivided accordingly, and special attention is given to the subclass $t_0 = 1$. Whereas there are no Λ -terms comprised in the class, the limit for vanishing vorticity leads to two previously derived irrotational dust families with $\Lambda > 0$, and the shear-free limit is the Gödel universe.

PACS numbers: 04.20.-q, 04.20.Jb, 04.40.Nr

1. INTRODUCTION

Dust space-times in general relativity, characterized as perfect fluids with vanishing pressure $p = 0$ and non-vanishing matter density $\mu \neq 0$, with or without cosmological constant Λ , have attracted a lot of attention. Whereas irrotational dust models form important arenas for studying both the late universe [1, 2] and gravitational collapse [3], exact rotating dust solutions may serve to describe phenomena on a galactic scale. Due to conservation of momentum the dust flow lines must be geodesics, and the remaining kinematic variables are the expansion scalar $\theta \equiv u^a_{;a}$, shear tensor $\sigma_{ab} \equiv u_{(a;b)} - \frac{1}{3}\theta h_{ab}$ ($\theta_{ab} \equiv u_{(a;b)}$ being the expansion tensor) and vorticity (or rotation) vector $\omega^a \equiv \frac{1}{2}\epsilon^{abc}u_{b;c}$, where u^a is the normalized dust 4-velocity, $h_{ab} \equiv g_{ab} + u_a u_b$ the projection tensor onto the comoving rest space, g_{ab} the space-time metric and $\epsilon_{abc} \equiv \eta_{abcd}u^d$ the spatial projection of the space-time permutation tensor η_{abcd} .

The Weyl tensor C_{abcd} , representing the locally free gravitational field, is fully determined by its electric part E_{ab} and magnetic part H_{ab} w.r.t. u^a , defined by

$$E_{ab} = C_{acbd} u^c u^d, \quad H_{ab} = \frac{1}{2}\epsilon_{amn} C^{mn}{}_{bd} u^d. \quad (1)$$

When $H_{ab} = 0, E_{ab} \neq 0$ the dust space-time is called *gravito-electric*, and the Petrov type is necessarily *I* or *D*. As the gravito-electric tensor is the general relativistic generalization of the tidal tensor in Newtonian theory (see e.g. [4]), such space-times were alternatively termed ‘Newtonian-like’ in [5].

For Newtonian-like irrotational dust models, the covariant propagation equations for σ_{ab} , θ , E_{ab} and μ , (equations (11-12) and (14-15) below, with $\omega^a = 0$) form an autonomous first order system, leading to a set of *ordinary* differential equations when projected onto the common eigenframe of $\sigma^a{}_b$ and $E^a{}_b$. No spatial gradients appear, such that each fluid element evolves as a separate universe, once the constraint equations are satisfied by the initial data. Such models were therefore termed

‘silent’ in [6]. The setting looked very appealing towards numerical schemes and simulations in astrophysical and cosmological context, e.g. for the description of structure formation in the universe and the study of the gravitational instability mechanism in general relativity [7, 8], where a clear motivation for taking $\omega^a = 0, H_{ab} \approx 0$ was given in [9].

However, in two independent papers [10, 11], the propagation of the constraint $H_{ab} = 0$ along u^a was shown to give rise to an infinite chain of integrability conditions and corresponding constraints on the above autonomous system. These constraints are identically satisfied for Petrov type *D*, but led the authors to conjecture that for the algebraically general case only orthogonally spatially homogeneous solutions of Bianchi type I (i.e., Saunders’ cosmological models [12]) would be allowed. In [13], however, the conjecture was shown to be false for strictly positive cosmological constant Λ , by the explicit construction of two solution families characterized by the presence of a geodesic space-like Weyl principal vector field. The generalized conjecture that these two families exhaust the inhomogeneous Newtonian-like ID models of Petrov type *I* was put forward in [14] [41].

The idea behind 1+3 covariantly silent models was extensively explained and deepened in the introduction of [10], while in the discussion section of the same paper weaker conditions than $H_{ab} = \omega^a = 0$ were indicated for establishing the silent property. One of them is to allow for vorticity. This is a natural generalization, since silent perfect fluids must have a vanishing spatial gradient of pressure [10] and hence are non-rotating *or* dust (as follows from the Frobenius theorem [16]). However, it was questioned at the same time whether this would appreciably broaden the class of silent solutions, as severe restrictions at first sight remain.

On the other hand, important classes of rotating dust models have been found by assuming some kind of symmetry, or are algebraically special. Respective examples are Winicour’s classification [17] of stationary axisymmetric models satisfying the circularity condition (see e.g. [18]), and the general rotating dust solution admitting time-like conformally flat hypersurfaces with zero extrinsic and constant intrinsic curvature as found by Stephani [19] and generalized by Barnes for non-zero

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Λ [20], which depends on seven free functions of one coordinate and which turns out to be gravito-electric and of Petrov type D . However, no algebraically general and asymmetric rotating dust solutions have so far been found [42].

The above serves as a clear motivation to investigate the class of Petrov type I , gravito-electric rotating dust models, which will be denoted by \mathcal{A} , in more detail. In this paper the main invariant properties of such models are deduced and a natural subdivision based on the number t_0 of functionally independent zero-order Riemann invariants is made. The paper is organized as follows. Section 2 introduces the 1+3 covariant and Weyl principal (WP) tetrad settings. The main result is contained in section 3, where it is proved that each member of \mathcal{A} necessarily has $\Lambda \neq 0$ and possesses a space-like geodesic Weyl principal vector field which is parallel to the vorticity, and where a completed set of algebraic relations between the basic scalar invariants (WP tetrad curvature components and connection coefficients) is presented. In section 4 further invariant properties are deduced and it is shown that $1 \leq t_0 \leq 2$. The subclass corresponding to $t_0 = 1$ is partitioned into two families: the first one comprises all non-expanding members of \mathcal{A} (for which necessarily $\Lambda < 0$), whereas the second one is characterized by equal norms of the vorticity and shear tensors (for which necessarily $\Lambda > 0$). Section 5 treats the zero vorticity, shear-free and Einstein space limit cases. The final section summarizes the results in a theorem and briefly discusses further features.

2. MATHEMATICAL SETTING

We use geometric units $8\pi G = c = 1$ and the signature $(-, +, +, +)$ for space-time metrics. Round (square) brackets denote (anti)symmetrization. By definition, a space-time belongs to \mathcal{A} if its metric g_{ab} is a solution of the field equation

$$R_{ab} - \frac{1}{2}R g_{ab} + \Lambda g_{ab} = \mu u_a u_b, \quad \mu \neq 0, \quad (2)$$

and if

$$H_{ab} = 0, \quad T_2^3 - T_3^2 \neq 0, \quad \omega^a \neq 0, \quad (3)$$

where

$$T_2 \equiv 6E^a{}_b E^b{}_a, \quad T_3 \equiv 36E^a{}_b E^b{}_c E^c{}_a \quad (4)$$

are convenient multiples of the quadratic and cubic Weyl invariants I and J [18].

In the 1+3 covariant approach [21, 22], the tensorial quantities ω_a , σ_{ab} , θ , E_{ab} and μ play the role of the fundamental dynamical fields, while the parameter Λ and the covariantly constant fields h_{ab} and ϵ_{abc} also enter the governing equations. A dot denotes covariant derivation

along u^a ('time propagation'). For convenience we define, for arbitrary tensors $T^{ab\dots}_{cd\dots}$ and for any natural number n , the operator \mathcal{O}_n by

$$\mathcal{O}_n(T)^{ab\dots}_{cd\dots} \equiv \dot{T}^{ab\dots}_{cd\dots} + n\theta T^{ab\dots}_{cd\dots}. \quad (5)$$

We further use the streamlined notation of [23]. The spatially projected, symmetric and trace-free part of a tensor S_{ab} is denoted by

$$S_{\langle ab \rangle} \equiv h_a{}^c h_b{}^d S_{cd} - \frac{1}{3} S_{cd} h^{cd} h_{ab} = 0. \quad (6)$$

The covariant spatial derivative D_a , acting on arbitrary tensors $T^{ab\dots}_{cd\dots}$, and the associated curl and divergence (div) operators, acting on one-tensors V_a and two-tensors S_{ab} , are defined by

$$D_e T^{ab\dots}_{cd\dots} = h^a{}_p h^b{}_q \dots h^r{}_c h^s{}_d \dots h^f{}_e T^{pq\dots}_{rs\dots}, \quad (7)$$

$$\text{div } V = D_a V^a, \quad \text{curl } V_a = \epsilon_{abc} D^b V^c, \quad (8)$$

$$\text{div } S_a = D^b S_{ab}, \quad \text{curl } S_{ab} = \epsilon_{cd(a} D^c S_{b)}{}^d. \quad (9)$$

The Ricci identity for u^a and the second Bianchi identity, both incorporating the field equation (2) via the substitution

$$R_{ab} = \left(\frac{\mu}{2} + \Lambda\right) h_{ab} + \left(\frac{\mu}{2} - \Lambda\right) u_a u_b, \quad (10)$$

are covariantly split into time propagation and constraint equations. For members of \mathcal{A} these are:

- Time propagation equations:

$$\mathcal{O}_{\frac{1}{3}}(\theta) = -\sigma_{ab}\sigma^{ab} + 2\omega^a\omega_a - \frac{\mu}{2} + \Lambda, \quad (11)$$

$$\mathcal{O}_{\frac{2}{3}}(\sigma)_{ab} = -\sigma_{c\langle a}\sigma_{b\rangle}{}^c - \omega_{\langle a}\omega_{b\rangle} - E_{ab}, \quad (12)$$

$$\mathcal{O}_{\frac{2}{3}}(\omega)_a = \sigma_{ab}\omega^b, \quad (13)$$

$$\mathcal{O}_1(E)_{ab} = 3\sigma_{c\langle a}E_{b\rangle}{}^c - \omega^c\epsilon_{cd\langle a}E_{b\rangle}{}^d - \frac{\mu}{2}\sigma_{ab}, \quad (14)$$

$$\mathcal{O}_1(\mu) = 0. \quad (15)$$

- Constraint equations:

$$\frac{2}{3}D_a\theta - \text{div } \sigma_a + \text{curl } \omega_a, \quad (16)$$

$$\text{curl } \sigma_{ab} + D_{\langle a}\omega_{b\rangle} = H_{ab}, \quad (17)$$

$$\text{div } \omega = 0, \quad (18)$$

$$\text{div } E_a - \frac{1}{3}D_a\mu = 0, \quad (19)$$

$$\text{curl } E_{ab} = 0, \quad (20)$$

$$\mathcal{C}^{(1)}{}_a \equiv 3E_{ab}\omega^b - [\sigma, E]_a + \mu\omega_a = 0, \quad (21)$$

where $[\sigma, E]_a \equiv \epsilon_{abc}\sigma^b{}_d E^{dc}$ is the one-tensor dual to the commutator of $\sigma^a{}_b$ and $E^a{}_b$.

Notice that the set of first order equations (11-15), augmented with $\dot{\Lambda} = \dot{h}_{ab} = \dot{\epsilon}_{abc} = 0$, indeed forms a 'silent' dynamical system, which is due to $\dot{u}^a = \text{curl } H_{ab} = 0$.

Expressing $\dot{H}_{ab} = 0$ and $\text{div } H_a = 0$, on the other hand, respectively yields the two constraints (20) and (21), the repeated time propagation of which gives rise to two chains of integrability conditions. In the next section attention will be focussed on the second one, leading to a completed set of invariant relations for \mathcal{A} under which the first chain turns out to be identically satisfied.

For a generic model contained in \mathcal{A} , let $\mathcal{B} \equiv (\partial_0^a = u^a, \partial_1^a, \partial_2^a, \partial_3^a)$ denote the WP tetrad, i.e., the essentially unique orthonormal eigenframe of E^a_b ($E_{12} = E_{13} = E_{23} = 0$). Below, capital Latin letters A, B, \dots denote WP tetrad indices and run from 0 to 3, while Greek letters α, β, \dots are $(\partial_1^a, \partial_2^a, \partial_3^a)$ -triad indices, run from 1 to 3 and have to be read ‘modulo 3’ (e.g. $\sigma_{\alpha+1\alpha-1} = \sigma_{12}$ for $\alpha = 3$). Einstein’s summation convention applies for both kinds of indices. In particular we have $h_{\alpha\beta} = \delta_{\alpha\beta}$, with δ the Kronecker delta symbol, and by convention we take $\epsilon_{123} = 1$ [22]. The action of a vector field X^a on a scalar function f is denoted by Xf . The commutator coefficients γ^A_{BC} and Ricci-rotation coefficients Γ^A_{BC} of \mathcal{B} are defined by

$$[\partial_B, \partial_C] = \gamma^A_{BC} \partial_A, \quad (\partial_B)^A{}_{;C} = \Gamma^A_{BC}. \quad (22)$$

As for any rigid frame, the lowered coefficients $\Gamma_{ABC} = g_{AD} \Gamma^D_{BC} = \Gamma_{[AB]C}$ and $\gamma_{ABC} = g_{AD} \gamma^D_{BC} = \gamma_{A[BC]}$ are biunivocally related by

$$\gamma_{ABC} = -2\Gamma_{A[BC]}, \quad \Gamma_{ABC} = -\gamma_{[AB]C} + \frac{1}{2}\gamma_{CAB}. \quad (23)$$

Within the orthonormal tetrad formalism w.r.t. \mathcal{B} , the matter density μ and suitable linear combinations of the eigenvalues E_α of E^a_b and of the rotation coefficients Γ_{ABC} play the role of invariantly defined basic variables. In the present paper we will use

$$\omega_\alpha = \Gamma_{0[\alpha-1\alpha+1]}, \quad (24)$$

$$\sigma_{\alpha+1\alpha-1} = \Gamma_{0(\alpha-1\alpha+1)}, \quad (25)$$

$$\theta = \Gamma^\beta_{0\beta}, \quad (26)$$

$$h_\alpha \equiv \sigma_{\alpha+1\alpha+1} - \sigma_{\alpha-1\alpha-1} \\ = \Gamma_{\alpha+10\alpha+1} - \Gamma_{\alpha-10\alpha-1}, \quad (27)$$

$$x_\alpha \equiv E_{\alpha+1} - E_{\alpha-1}, \quad (28)$$

where $h_1 = -(h_2 + h_3)$ and $x_1 = -(x_2 + x_3)$, together with [43]

$$\Omega_\alpha \equiv \Gamma_{\alpha-1\alpha+10}, \quad (29)$$

$$n_\alpha \equiv \Gamma_{\alpha+1\alpha-1\alpha}, \quad (30)$$

$$q_\alpha \equiv -\Gamma_{\alpha\alpha-1\alpha-1} = \gamma_{\alpha-1\alpha-1\alpha} \quad (31)$$

$$r_\alpha \equiv \Gamma_{\alpha\alpha+1\alpha+1} = \gamma_{\alpha+1\alpha\alpha+1}. \quad (32)$$

In terms of the x_α the quadratic and cubic Weyl invariants read

$$T_2 = 4(x_2^2 + x_2x_3 + x_3^2), \quad T_3 = -4(x_1 - x_2)(x_2 - x_3)(x_3 - x_1) \quad (33)$$

and the Petrov type *I* condition $T_2^3 - T_3^2 \neq 0$ becomes $x_1x_2x_3 \neq 0$.

The basic equations of the formalism are the commutator relations, i.e., the first part of (22) applied to scalar functions f (further denoted by $\text{com}_{BC} f$), and the projected Ricci and second Bianchi equations. Within the ON formalism the Ricci equations are further split into:

1. the first Bianchi equations $R_{A[BCD]} = 0$, equivalent to the Jacobi identities

$$[\partial_A, [\partial_B, \partial_C]]^D = \partial_A \gamma^D_{BC} + \gamma^F_{[BC} \gamma^D_{A]F} = 0. \quad (34)$$

Here $[\partial_0, [\partial_{\alpha+1}, \partial_{\alpha-1}]]^0 = 0$ is the α -component of (13), while $[\partial_1, [\partial_2, \partial_3]]^0 = 0$ is (18);

2. the $\alpha\beta$ -components of (12) and (17);

3. the tetrad components of the field equation (2). Here the 00-component is Raychaudhuri’s equation (11) and the 0α -component is (16); the $\alpha\beta$ -components

$$\partial_C \Gamma^C_{(\alpha\beta)} - \partial_{(\beta} \Gamma^C_{\alpha)C} + \Gamma^C_{DC} \Gamma^D_{(\alpha\beta)} - \Gamma^C_{\alpha D} \Gamma^D_{\beta C} \\ = \left(\frac{\mu}{2} + \Lambda \right) \delta_{\alpha\beta} \quad (35)$$

are not covered by the Ricci-identity for u^a .

The formulas

$$\dot{V}_\alpha = \partial_0 V_\alpha + \epsilon_{\alpha\beta\gamma} \Omega^\beta V^\gamma, \quad (36)$$

$$\dot{S}_{\alpha\beta} = \partial_0 S_{\alpha\beta} + 2\epsilon_{\gamma\delta(\alpha} \Omega^\gamma S_{\beta)}^\delta, \quad (37)$$

$$D_\alpha V_\beta = \partial_\alpha V_\beta - V_\delta \Gamma^\delta_{\beta\alpha}, \quad (38)$$

$$D_\alpha S_{\beta\gamma} = \partial_\alpha S_{\beta\gamma} - 2S_{\delta(\gamma} \Gamma^\delta_{\beta)\alpha} \quad (39)$$

and definitions (8-9) relate the covariant differential operations to directional derivatives.

3. MAIN RESULT

Starting from $\mathcal{C}^{(1)}_a = 0$ one subsequently derives the necessary conditions

$$\mathcal{C}^{(2)}_a \equiv \mathcal{O}_{\frac{5}{3}}(\mathcal{C}^{(1)})_a + \frac{1}{2}\sigma_{ab}\mathcal{C}^{(1)b} + \frac{1}{6}\epsilon_{abc}\omega^b\mathcal{C}^{(1)c} \\ \equiv \frac{8}{3}\sigma_a{}^b E_b{}^c \omega_c + \frac{10}{3}E_a{}^b \sigma_b{}^c \omega_c - \sigma_b{}^c E_c{}^b \omega_a = 0$$

and

$$\mathcal{C}^{(3)}_a \equiv \mathcal{O}_{\frac{7}{3}}(\mathcal{C}^{(1)})_a \\ \equiv \frac{4}{3}\sigma_a{}^b \sigma_b{}^c E_c{}^d \omega_d + \frac{35}{3}\sigma_a{}^b E_b{}^c \sigma_c{}^d \omega_d + 5E_a{}^b \sigma_b{}^c \sigma_c{}^d \omega_d \\ - 2\sigma_b{}^c E_c{}^d \sigma_d{}^b \omega_a - 7\sigma_c{}^d E_d{}^c \sigma_a{}^b \omega_b + 2\sigma_c{}^d \sigma_d{}^c E_a{}^b \omega_b \\ - \frac{4}{3}\omega^c \omega_c E_a{}^b \omega_b - \frac{5}{3}E_{bc}\omega^b \omega^c \omega_a - 6E_a{}^b E_b{}^c \omega_c \\ + E_b{}^c E_c{}^b \omega_a - 3\mu\sigma_a{}^b \sigma_b{}^c \omega_c + \frac{\mu}{2}\sigma_b{}^c \sigma_c{}^b \omega_a \\ - \frac{5}{3}\epsilon_{abc}\omega^b E_c{}^d \sigma_d{}^e \omega_e + \frac{5}{3}E_a{}^b \epsilon_{bcd}\omega^c \sigma_d{}^e \omega_e \\ - \frac{4}{3}\sigma_a{}^b \epsilon_{bcd}\omega^c E_d{}^e \omega_e + \omega^b \epsilon_{bcd} E_c{}^e \sigma^e{}^d \omega_a = 0.$$

The validity of these equations has been independently checked in an *unspecified* tetrad approach, hereby using the Maple computer algebra package. Note that the expansion scalar θ does not enter the expressions, which is a great technical advantage in view of later elimination processes.

In this section we will show that the chain of algebraic integrability conditions generated by further time propagation of $\mathcal{C}^{(3)}_a = 0$ *terminates*. We will eventually describe its complete solution set both covariantly and in terms of WP tetrad invariants. This will be achieved by a number of propositions.

Denote

$$F_\alpha \equiv h_{\alpha+1}x_{\alpha-1} + h_{\alpha-1}x_{\alpha+1}, \quad (40)$$

$$Z_\alpha \equiv \mu(x_{\alpha+1} - x_{\alpha-1}) + 2(x_{\alpha+1}^2 + x_{\alpha-1}^2). \quad (41)$$

Projection of $\mathcal{C}^{(1)}_a = 0$ w.r.t. the WP tetrad gives

$$\sigma_{\alpha+1\alpha-1} = -\frac{\mu - x_{\alpha+1} + x_{\alpha-1}}{x_\alpha} \omega_\alpha, \quad (42)$$

and substituting this in the components of $\mathcal{C}^{(2)}_a = 0$ yields

$$\omega_1 x_2 x_3 F_1 + 2\omega_2 \omega_3 x_1 Z_1 = 0, \quad (43)$$

$$\omega_2 x_3 x_1 F_2 + 2\omega_3 \omega_1 x_2 Z_2 = 0, \quad (44)$$

$$\omega_3 x_1 x_2 F_3 + 2\omega_1 \omega_2 x_3 Z_3 = 0. \quad (45)$$

The following lemmas allow to draw quick conclusions in later proofs. Lemma 3.1 especially helps to avoid explicit calculations; for its proof we need (15) together with the diagonal components of (14), namely

$$\mathcal{O}_1(x_2) = \left(x_2 + x_3 - \frac{\mu}{2}\right) h_2 + x_2 h_3, \quad (46)$$

$$\mathcal{O}_1(x_3) = -x_3 h_2 - \left(x_2 + x_3 + \frac{\mu}{2}\right) h_3. \quad (47)$$

Lemma 3.1 Suppose x_2, x_3 and μ are constrained by two relations

$F(x_2, x_3, \mu) = G(x_2, x_3, \mu) = 0$, where F and G are homogeneous polynomials with integer coefficients and without common factors. Then either $h_2 = h_3 = 0$ or $\mu^2 = T_2$.

Proof. As F and G do not have a common factor, their resultant w.r.t. x_3 is non-zero [44] and thus leads to at least one *irreducible* homogeneous polynomial relation $P(x_2, \mu) = 0$. Let n be the total degree of P . As $\mathcal{O}_1(\mu) = 0$, we obtain $\mathcal{O}_n(P(x_2, \mu)) = \frac{\partial P}{\partial x_2}(x_2, \mu) \mathcal{O}_1(x_2) = 0$. Now $\frac{\partial P}{\partial x_2}$ and P are both homogeneous and cannot have a common factor, since $\frac{\partial P}{\partial x_2}$ has a strictly lower degree than P and P is irreducible. Hence $\frac{\partial P}{\partial x_2}(x_2, \mu) = P(x_2, \mu) = 0$ would lead to $(x_2, \mu) = (0, 0)$, which is excluded. Thus $\mathcal{O}_1(x_2) = 0$, and an analogous reasoning based on the resultant of F and G w.r.t. x_2 yields $\mathcal{O}_1(x_3) = 0$. By (46-47) this gives a system of two linear and homogeneous equations in (h_2, h_3) . Hence either $h_2 = h_3 = 0$ or the determinant of the system matrix, computed to be $(\mu^2 - T_2)/4$, vanishes. \square

Lemma 3.2 (a) If two F_β 's vanish at the same time, then $h_2 = h_3 = 0$.

(b) Two Z_β 's cannot vanish at the same time.

(c) If, for fixed β , $Z_\beta = 0$ then $h_2 = h_3 = 0$.

Proof. By cyclicity it is sufficient to prove (a) for $F_2 = F_3 = 0$, (b) for $Z_2 = Z_3 = 0$ and (c) for $Z_1 = 0$.

(a) $F_2 = F_3 = 0$ forms a linear and homogeneous system in the variables (h_2, h_3) , the determinant of which is constantly proportional to x_1^2 and hence cannot vanish. Thus $h_2 = h_3 = 0$.

(b) Elimination of μ from $Z_2 = Z_3 = 0$ yields $x_1(4x_2^2 + 7x_2x_3 + 4x_3^2) = 0$, contradictory to the Petrov type *I* assumption.

(c) One first calculates that

$$\begin{aligned} \mathcal{O}_2(Z_1) &= (4x_2^2 - 4x_3^2 + 4x_2x_3 + \mu(2x_3 - x_2) - \mu^2/2)h_2 \\ &\quad + (4x_2^2 - 4x_3^2 - 4x_2x_3 + \mu(2x_2 - x_3) + \mu^2/2)h_3. \end{aligned}$$

Combining the assumptions $Z_1 = 0$ and $\omega^a \neq 0$ with the equations (43-45) it follows that $F_1 F_2 F_3 = 0$. Now suppose that $(h_2, h_3) \neq (0, 0)$. Then, for fixed k , the determinant $D_k = D_k(x_2, x_3, \mu)$ of the linear and homogeneous system $\mathcal{O}_2(Z_1) = F_k = 0$ in the variables (h_2, h_3) should vanish. As the computed D_k is not a multiple of Z_1 , lemma 3.1 applies with $F = Z_1$ and $G = D_k$. Because of the hypothesis only the possibility $\mu^2 - T_2 = 0$ remains, but elimination of μ from this equation and $Z_1 = 0$ yields $x_1^2 x_2 x_3 = 0$, contradictory to the Petrov type *I* assumption. Thus $h_2 = h_3 = 0$. \square

The key step in the deduction is the following

Proposition 3.3 $\omega_1 \omega_2 \omega_3 = 0$.

Proof. Suppose on the contrary that $\omega_1 \omega_2 \omega_3 \neq 0$. Firstly, if h_2 and h_3 both vanished, one would have that all $F_\alpha = 0$ and hence, by (43-45), all $Z_\alpha = 0$, which is impossible according to lemma 3.2(b). Thus $(h_2, h_3) \neq (0, 0)$, whence also $Z_1 Z_2 Z_3 \neq 0$ by lemma 3.2(c). Secondly, the equations (44), (45) form a linear and homogeneous system in the variables (ω_2, ω_3) , the determinant of which must vanish by the hypothesis. Making analogous observations for the couples (43), (44) and (43), (45) one arrives at

$$L_\alpha \equiv 4Z_{\alpha+1}Z_{\alpha-1}\omega_\alpha^2 - F_{\alpha+1}F_{\alpha-1}x_\alpha^2 = 0. \quad (48)$$

The key steps are now the following. Substituting (42) into the first component of $\mathcal{C}^{(3)}_a = 0$ one obtains an equation of the form

$$(P_1\omega_1^2 + P_2\omega_2^2 + P_3\omega_3^2 + Qx_1^2x_2^3x_3^3)\omega_1 - Rx_1\omega_2\omega_3 = 0 \quad (49)$$

where P_1, P_2, P_3 are homogeneous polynomials in (x_2, x_3, μ) of total degree 7, while Q and R are homogeneous in (x_2, x_3, μ) and quadratic, resp. linear homogeneous in (h_2, h_3) . Now multiply (49) with $2Z_1$, add RL_1 , divide the result by ω_1 and finally eliminate the

ω_α^2 's by means of (48). Performing the analogous operations on the second and third components of $\mathcal{C}^{(3)}_a = 0$ one derives three equations $G_\alpha = 0$, where

$$G_\alpha \equiv Q_{\alpha+1\alpha+1}h_2^2 + Q_{\alpha+1\alpha-1}h_2h_3 + Q_{\alpha-1\alpha-1}h_3^2 + 2Z_1Z_2Z_3,$$

Q_{22}, Q_{23} and Q_{33} being homogeneous polynomials in (x_2, x_3, μ) of total degree 5. Consistency of these equations with $(h_2, h_3) \neq (0, 0)$ and $Z_1Z_2Z_3 \neq 0$ requires that $F = 0$, where

$$F \equiv -1107\mu^8 + T_2\mu^6 + 85T_3\mu^5 - 1510T_2^2\mu^4 - 63T_2T_3\mu^3 + 2(130T_2^3 - 3T_3^2)\mu^2 - 14T_3T_2^2\mu - 2T_2(4T_2^3 - T_3^2),$$

as is readily deduced by putting h_2 equal to 1 in $G_2 - G_1$ and $G_3 - G_1$, and then computing the resultant w.r.t. h_3 of the resulting polynomials, which yields $Z_1Z_2Z_3F = 0$. By repeating the same procedure for $G_2 - G_1$ and $\mathcal{O}_8(F)$, which is linear homogeneous in (h_2, h_3) by (15) and (46-47), one arrives at a relation $G(x_2, x_3, \mu) = 0$, where G is a homogeneous polynomial of total degree 18 which is not a multiple of F . With this F and G one deduces from lemma 3.1 that $\mu^2 - T_2 = 0$. Calculating the resultant of $\mu^2 - T_2$ and F , however, one finds $T_2^2(T_2^3 - T_3^2)^2 = 0$ such that the Petrov type is O or D, which yields the desired contradiction. \square

Proposition 3.4 The vorticity is parallel to a Weyl principal vector, i.e., it is an eigenvector of E^a_b .

Proof. By proposition 3.3, we must show that the case $\omega_1 = 0, \omega_2\omega_3 \neq 0$ is inconsistent. From this hypothesis and (43) we get $Z_1 = 0$. Hence, by (41) with $\alpha = 1$, $x_2 \neq x_3$ and

$$\mu = 2\frac{x_2^2 + x_3^2}{x_2 - x_3}. \quad (50)$$

The $\alpha = 1$ -component of (42) immediately yields $\sigma_{23} = 0$, while substitution of (50) into the 2- and 3-components gives

$$\sigma_{12} = \omega_3 \frac{x_2 + 3x_3}{x_2 - x_3}, \quad \sigma_{13} = \omega_2 \frac{3x_2 + x_3}{x_2 - x_3}, \quad (51)$$

respectively. Likewise, the 23-component of (14) yields $\Omega_1 = 0$, while substitution of (50) into the 31- and 12-components gives

$$\Omega_2 = 4\omega_2x_2 \frac{x_2 + x_3}{(x_2 - x_3)^2}, \quad \Omega_3 = 4\omega_3x_3 \frac{x_2 + x_3}{(x_2 - x_3)^2}, \quad (52)$$

such that $\Omega_2\Omega_3 \neq 0$. Next, by (44-45) and lemma 3.2(a), or by $Z_1 = 0$ and lemma 3.2(c), we conclude that $h_2 = h_3 = 0$. Propagating this along the ∂_0^a integral curves, on using the diagonal components of (12), and then substituting (51) and (52) one gets

$$\omega_2^2 = \frac{(x_2 + 2x_3)(x_2 - x_3)^3}{48x_2(x_2 + x_3)^2}, \quad \omega_3^2 = \frac{(2x_2 + x_3)(x_2 - x_3)^3}{48x_3(x_2 + x_3)^2}, \quad (53)$$

such that $(x_2 + 2x_3)(2x_2 + x_3) \neq 0$. Applying \mathcal{O}_1 on the left and right hand sides of these two equations one finds $\theta\omega_2^2 = \theta\omega_3^2 = 0$, whence $\theta = 0$. With the so far obtained equations Raychaudhuri's equation (11) reduces to

$$2x_2x_3 + \Lambda(x_2 - x_3) = 0. \quad (54)$$

Using (50), (52), (53) and (54) one derives the simple relations

$$\mu = 2\Lambda + 2(x_3 - x_2), \quad (55)$$

$$x_2 - x_3 = 3(\Omega_2^2 + \Omega_3^2 + \Lambda), \quad (56)$$

$$x_2 + x_3 = 3(\Omega_2^2 - \Omega_3^2). \quad (57)$$

At this stage (12)-(15) reduces to the vanishing of $\sigma_{13}, \sigma_{12}, \omega_2, \omega_3, x_2, x_3$ and μ under ∂_0 , such that the above algebraic relations either propagate consistently along the ∂_0^a integral curves, in a trivial way, or lead to $\partial_0\Omega_2 = \partial_0\Omega_3 = 0$. However, propagating them along the ∂_a^a integral curves will lead to a contradiction. Doing this for (55) and mixing up with equation (19) and the off-diagonal components of (20) leads to

$$\partial_1x_2 = 2r_1x_3 + q_1x_2, \quad \partial_1x_3 = -r_1x_3 - 2q_1x_2, \quad (58)$$

$$\partial_2x_2 = -r_2x_1 + q_2x_3, \quad \partial_2x_3 = -r_2x_1 - \frac{q_2x_3}{2}, \quad (59)$$

$$\partial_3x_2 = q_3x_1 + \frac{r_3x_2}{2}, \quad \partial_3x_3 = q_3x_1 - r_3x_2. \quad (60)$$

The diagonal components of (20) on the other hand reduce to

$$(x_2 + x_3)n_1 + x_2n_2 = 0, \quad (x_2 + x_3)n_1 + x_3n_3 = 0. \quad (61)$$

Two more algebraic relations follow by suitably combining the 2- and 3-components of (16) with the 31- and 12-components of (17), respectively, and substituting (51), namely

$$\omega_3(x_2 + x_3)q_1 + \omega_2(3x_2 + x_3)n_2 + 2\omega_2x_2n_3 = 0, \quad (62)$$

$$\omega_2(x_2 + x_3)r_1 + 2\omega_3x_3n_2 + \omega_3(x_2 + 3x_3)n_3 = 0. \quad (63)$$

Propagation of (54) along the ∂_1^a integral curves leads to

$$x_2(2x_2^2 + x_3^2)q_1 + x_3(x_2^2 + 2x_3^2)r_1 = 0. \quad (64)$$

The equations (61)-(64) form a linear and homogeneous system in q_1, r_1 and the n_α . Calculating the determinant of the system matrix and substituting (53) one finds $(x_2 - x_3)^3(x_2 + 2x_3)(2x_2 + x_3)(x_2 + x_3)(x_2^2 + x_3^2)/16$, which cannot vanish. Hence

$$q_1 = r_1 = n_1 = n_2 = n_3 = 0. \quad (65)$$

Next, applying ∂_2 and ∂_3 to (54) leads to

$$x_3(x_2^2 + 2x_3^2)q_2 + 2(x_3 - x_2)(x_2 + x_3)^2r_2 = 0, \quad (66)$$

$$2(x_3 - x_2)(x_2 + x_3)^2q_3 - x_2(2x_2^2 + x_3^2)r_3 = 0, \quad (67)$$

respectively. The $\text{com}_{01}(x_2 - x_3)$ commutator relation becomes

$$\omega_3 x_3^3 q_2 - \omega_2 x_2^3 r_3 = 0. \quad (68)$$

The $\text{com}_{01}(x_2 + x_3)$ commutator relation is a combination of (66), (67) and (68). However, by propagating the first equation of (53) along ∂_2^a and the second one along ∂_3^a we find

$$\begin{aligned} \partial_2 \omega_2 &= \frac{(x_2 - x_3)^2}{192(x_2 + x_3)^3 x_2^2 \omega_2} \times \\ &\quad (x_3(5x_2 + x_3)(x_2 + 2x_3)^2 q_2 - 4(x_2^2 + 4x_2 x_3 + x_3^2) r_2), \\ \partial_3 \omega_3 &= \frac{(x_2 - x_3)^2}{192(x_2 + x_3)^3 x_3^2 \omega_3} \times \\ &\quad (4(x_2^2 + 4x_2 x_3 + x_3^2) q_3 + x_3(x_2 + 5x_3)(2x_2 + x_3)^2 r_3), \end{aligned}$$

respectively. Substituting this in (18) one gets a new independent relation which, together with (66-68), forms a linear and homogeneous system in q_2 , r_2 , q_3 and r_3 . After substitution of (53) the determinant of the system matrix is a homogeneous polynomial in (x_2, x_3) . If it does not vanish then $q_2 = r_2 = q_3 = r_3 = 0$; if it does vanish then it follows in combination with (54) that x_2 and x_3 are constant. Looking at (59) and (60) we conclude that in any case

$$q_2 = r_2 = q_3 = r_3 = 0. \quad (69)$$

Finally, substitution of (51-53), (55), (65), (69) and $\theta = h_2 = h_3$ into the 11-component of (35) yields $x_2 - x_3 = 3\Lambda$ and thus, by comparison with (56), $\Omega_2^2 + \Omega_3^2 = 0$, which is in contradiction with $\Omega_2 \Omega_3 \neq 0$. \square

By proposition 3.4 the vorticity vector is parallel to e.g. ∂_1^a , such that $\omega_1 \neq 0$ and

$$\omega_2 = \omega_3 = 0. \quad (70)$$

Then the 2- and 3-components of (42) simply read

$$\sigma_{12} = \sigma_{13} = 0 \quad (71)$$

and the 31- and 12-components of (14) yield

$$\Omega_2 = \Omega_3 = 0. \quad (72)$$

With these specifications, the equations (44-45) are automatically satisfied, whereas (43) gives $F_1 = 0$, i.e.,

$$h_2 x_3 + h_3 x_2 = 0. \quad (73)$$

Now the 1-components of $\mathcal{C}^{(1)}_a$ and $\mathcal{C}^{(3)}_a/(12\omega_1)$ are of the form

$$\mu \omega_1 + F_1, \quad \mu(h_2 h_3 + \sigma_{23}^2) + F_2, \quad (74)$$

where F_1 and F_2 do not contain μ . Calculating one further derivative $\mathcal{O}_3(\mathcal{C}^{(3)}_a)$ and dividing the 1-component by $2\omega_1$ one gets a condition of the form

$$(h_3 - h_2)[4(h_2 h_3 + \sigma_{23}^2) + 9\omega_1^2 - 12(h_2 x_3 + h_3 x_2)]\mu + F_3 = 0,$$

with F_4 again independent of μ , which thus can be obviously eliminated by means of (73) and (74). But doing so one miraculously arrives at $(h_2 x_3 + h_3 x_2)F_4 + x_2 x_3(h_3 - h_2 + \theta) = 0$, such that

$$\theta = h_2 - h_3 \quad \text{i.e.} \quad \theta_{11} = 0. \quad (75)$$

by the Petrov type I assumption. Thus the restriction to u^\perp of the expansion tensor θ^a_b has a zero eigenvalue, $\theta_1 = \sigma_1 + \theta/3 = 0$, as is seen from the representation matrix

$$[\theta_{\alpha\beta}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -h_3 & \sigma_{23} \\ 0 & \sigma_{23} & h_2 \end{bmatrix}, \quad (76)$$

w.r.t. the WP triad $(\partial_1^a, \partial_2^a, \partial_3^a)$. Applying ∂_0 to (75) one gets

$$\mu = 2(x_2 - x_3 + \Lambda) \quad \text{i.e.} \quad \mu = -6E_1 + 2\Lambda. \quad (77)$$

The ∂_2 - and ∂_3 -derivatives hereof, together with $x_2 x_3 \neq 0$, give

$$q_2 = r_3 = 0. \quad (78)$$

Together with (75) this implies that *the space-times possess a geodesic space-like WP vector field*, in casu ∂_1^a . In combination with (77) the 23-components of (14) and (42) yield

$$\sigma_{23} = -\frac{x_2 - x_3 + 2\Lambda}{x_1} \omega_1, \quad \Omega_1 = -2\frac{(x_2 + \Lambda)(x_3 - \Lambda)}{x_1^2} \omega_1. \quad (79)$$

A further ∂_0 -derivative of (77) leads us back to (73). Propagating (73) along ∂_0^a and substituting (79) gives

$$8\Lambda\omega_1^2(x_2 + \Lambda)(x_3 - \Lambda) - 2x_2 x_3(x_1^2 - \Lambda\theta^2) = 0. \quad (80)$$

On using (73) and the first equation of (79) this may still be simplified to

$$(h_2 h_3 + \sigma_{23}^2 - \omega_1^2)\Lambda + x_2 x_3 = 0. \quad (81)$$

As the Petrov type is assumed to be I , (80) or (81) implies that $\Lambda \neq 0$. The ∂_0 -derivative of (81) is found to be identically satisfied under the already derived equations.

Equivalently, one may start with the *covariant* expression of (75), namely

$$\theta_{ab}\omega^b = 0 \quad \text{i.e.} \quad \sigma_{ab}\omega^b = -\frac{\theta}{3}\theta\omega_a, \quad (82)$$

with $\omega^a \neq 0$. Two covariant time derivatives lead to

$$E_{ab}\omega^b = -\frac{1}{6}(\mu - 2\Lambda)\omega_a, \quad (83)$$

$$\text{tr}(\sigma E) = \frac{\theta}{3}(\mu - 2\Lambda), \quad (84)$$

which, given (82), is nothing but (73) and (77). Now $\mathcal{C}^{(1)}_a = 0$ simplifies to

$$[\sigma, E]_a = \frac{1}{2}(\mu + 2\Lambda)\omega_a, \quad (85)$$

the covariant time derivative of which may be reduced by the already obtained equations to

$$\begin{aligned} \text{tr } E^2 - \frac{1}{6}(\mu - 2\Lambda)^2 - 2\text{tr}(\sigma^2 E) + \frac{2}{3}\theta\text{tr}(\sigma E) \\ + \frac{1}{18}(\mu + 4\Lambda)(3\text{tr } \sigma^2 - 2\theta^2) - 2\Lambda\omega^a\omega_a = 0, \end{aligned} \quad (86)$$

The time propagation of (86) onto the WP tetrad is identically satisfied under the projections of (82-86).

We conclude that, for perfect fluids satisfying (2-4), *the complete set of covariant equations generated by time propagation of (21) is equivalent to (82-86) and, after projection onto the Weyl principal tetrad, to (70-73), (75), (77), (79) and (81).*

4. FURTHER INVARIANT PROPERTIES AND SUBDIVISION

From here it is advantageous to use

$$U_1 \equiv \Gamma_{203} = \omega_1 + \sigma_{23}, \quad V_1 \equiv -\Gamma_{302} = \omega_1 - \sigma_{23} \quad (87)$$

as auxiliary variables. This allows to rewrite (79) as

$$(x_3 - \Lambda)U_1 = (x_2 + \Lambda)V_1 = x_1\Omega_1 \equiv -[(x_2 + \Lambda) + (x_3 - \Lambda)]\Omega_1, \quad (88)$$

which is a linear and homogeneous system in the variables $(x_2 + \Lambda, x_3 - \Lambda)$. As we would get $x_1 = -(x_2 + x_3) = 0$ if both these variables were zero, the determinant of the system should vanish:

$$U_1V_1 + U_1\Omega_1 + V_1\Omega_1 = 0, \quad \text{i.e.} \quad \Omega_1 = \frac{\sigma_{23}^2 - \omega_1^2}{2\omega_1}. \quad (89)$$

After some manipulation, the projections of the Bianchi identity and Ricci identity for u^a reduce to

$$\partial_0 x_2 = -h_2(x_2 + \Lambda), \quad \partial_0 x_3 = h_3(x_3 - \Lambda), \quad (90)$$

$$\partial_0 h_2 = -h_2^2 - 2\Omega_1 U_1 - x_2, \quad (91)$$

$$\partial_0 h_3 = h_3^2 + 2\Omega_1 V_1 - x_3, \quad (92)$$

$$\partial_0 \omega_1 = -\theta\omega_1, \quad \partial_0 \sigma_{23} = -\Omega_1 h_1 - \theta\sigma_{23}, \quad (93)$$

$$\partial_1 x_2 = -q_1 x_2, \quad \partial_1 x_3 = r_1 x_3, \quad (94)$$

$$\partial_2 x_2 = -r_2 x_1, \quad \partial_3 x_3 = q_3 x_1, \quad (95)$$

$$\partial_1 h_2 = -q_1 h_2 - 2n_1 \sigma_{23} + n_3 V_1, \quad (96)$$

$$\partial_1 h_3 = r_1 h_3 - 2n_1 \sigma_{23} - n_2 U_1, \quad (97)$$

$$\partial_1 U_1 = -q_1 U_1 + 2n_1 h_2, \quad \partial_1 V_1 = r_1 V_1 - 2n_1 h_3, \quad (98)$$

$$\partial_2 h_2 + \partial_3 V_1 = -r_2 h_1 + 2q_3 \sigma_{23}, \quad (99)$$

$$\partial_3 h_3 + \partial_2 U_1 = q_3 h_1 + 2r_2 \sigma_{23}, \quad (100)$$

$$q_1 V_1 - r_1 U_1 + 2n_1(h_2 - h_3) = 0, \quad (101)$$

$$n_1 x_1 = n_2 x_2 = n_3 x_3 = 0, \quad (102)$$

while the 11-component and subtraction of the 22- and 33-components of (35) yield

$$\partial_1 q_1 = -q_1^2 - 2n_1 n_3 - (x_2 + \Lambda), \quad (103)$$

$$\partial_1 r_1 = r_1^2 + 2n_1 n_2 - (x_3 - \Lambda). \quad (104)$$

On calculating the covariant time derivative of the remaining primary integrability condition (20), one checks that its projection w.r.t. the WP tetrad is identically satisfied under the above equations. For this purpose one uses the ‘curl-dot’ 1+3 covariant commutator relation applied to the two-tensor E_{ab} , which in our situation becomes [45]

$$\begin{aligned} (\text{curl } S)_{ab} = \text{curl } \dot{S}_{ab} - \frac{1}{3}\theta \text{curl } S_{ab} - \sigma_e^c \epsilon_{cd(a} D^e S_{b)}^d \\ - D_{(a} S_{b)}^d \omega_d + \omega_{(a} \text{div } S_{b)}. \end{aligned}$$

Further properties of \mathcal{A} may be deduced. Firstly, $[\sigma, E]_a = 0 (\Leftrightarrow \sigma_{23} = 0)$ is impossible: $\sigma_{23} = 0 \neq \omega_1$, (93) and (89) would give $-h_1 \equiv h_2 + h_3 = 0$, and then (91-92) would yield $x_1 = 0$ and hence Petrov type D . Secondly, the shear tensor, or equivalently the expansion tensor, cannot be degenerate. As $\sigma_{23} \neq 0$, this would imply $h_3 h_3 + \sigma_{23}^2 = 0$, which expresses the vanishing of the determinant of the non-trivial 2 by 2 block in (76). But taking the ∂_0 -derivative hereof and using (73), (89) and $x_2 x_3 \neq 0$ would then yield $h_2 = h_3 = \sigma_{23} = 0$, a contradiction. Thirdly, (77) implies that for each member of \mathcal{A} the number t_0 of functionally independent zero-order Riemann invariants is at most 2. The following proposition states this much more precisely.

Proposition 4.1 Within \mathcal{A} one has $1 \leq t_0 \leq 2$, with moreover $t_0 = 1$ if and only if one of the three mutually exclusive possibilities $\theta = 0$, $U_1 = 0$ or $V_1 = 0$ is satisfied. The cosmological constant is positive for $U_1 V_1 \equiv \omega_1^2 - \sigma_{23}^2 = 0$ and negative for $\theta = 0$.

PROOF. We have $t_0 \leq 1$ if and only if the differentials dx_2 and dx_3 are algebraically dependent at each point. Then, in particular,

$$\partial_0 x_2 \partial_1 x_3 - \partial_1 x_2 \partial_0 x_3 = 0. \quad (105)$$

From (73), (90), (94) and $x_2 x_3 \neq 0$ it follows that this equation is identically satisfied in the case where $\theta = h_2 - h_3 = 0$ (which implies $h_2 = h_3 = 0$), and that it gives

$$(x_3 - \Lambda)q_1 + r_1(x_2 + \Lambda) = 0 \quad (106)$$

when $\theta \neq 0$. In combination with (88) and $x_1 \neq 0$, (106) is equivalent to

$$q_1 V_1 + r_1 U_1 = 0. \quad (107)$$

Now equations (73), (81) and the first equation of (88) may be solved for x_2 , x_3 and Λ , giving

$$x_2 = -\frac{FG}{2h_3\omega_1}, \quad x_3 = \frac{FG}{2h_2\omega_1}, \quad \Lambda = \frac{FG^2}{4h_2h_3\omega_1^2}, \quad (108)$$

where

$$F \equiv h_2 h_3 - U_1 V_1 \neq 0, \quad G \equiv h_2 V_1 + h_3 U_1 \neq 0.$$

From (73) and (101-102) one obtains

$$n_2 = \frac{q_1 V_1 - r_1 U_1}{2h_2}, \quad n_3 = -\frac{q_1 V_1 - r_1 U_1}{2h_3}, \quad (109)$$

$$n_1 = \frac{q_1 V_1 - r_1 U_1}{2(h_2 - h_3)}. \quad (110)$$

Taking the ∂_1 -derivative of (107), using (98) and (103-104), and substituting (108) and (109-110), one surprisingly finds

$$U_1 V_1 F G \frac{4\omega_1^2 r_1^2 + \theta^2 V_1^2}{2h_2 h_3 \theta \omega_1^2} = 0. \quad (111)$$

Together with (108) and $\theta x_2 x_3 \neq 0$ it follows that $U_1 V_1 = 0$.

Conversely, if $\theta = 0$, i.e., $h_2 = h_3 = 0$, then (91-92) implies $x_2 = -2\Omega_1 U_1$ and $x_3 = 2\Omega_1 V_1$, such that in particular $U_1 V_1 \Omega_1 \neq 0$. Inserting this into (81) one finds that $\Lambda = -4\Omega_1^2 < 0$; whence, without loss of generality,

$$\lambda \equiv \sqrt{-\Lambda}, \quad \Omega_1 = \frac{\lambda}{2}, \quad U_1 = -\frac{x_2}{\lambda}, \quad V_1 = \frac{x_3}{\lambda}. \quad (112)$$

Herewith, either of the equations (88) becomes

$$2x_2 x_3 + (x_2 - x_3)\lambda^2 = 0, \quad (113)$$

which establishes that the zero-order Riemann invariants are *algebraically* dependent. On the other hand, if e.g. $V_1 = 0$ ($\sigma_{23} = \omega_1$) then $U_1 \neq 0$, and we consecutively deduce $x_3 = \Lambda$ from (88), $x_2 = -h_2 h_3$ from (81) and $\Lambda = h_2^2 > 0$ from (73). The case $U_1 = 0$ is equivalent to $V_1 = 0$ (switching of 2- and 3-axes) and leads to $x_2 = -\Lambda$, $x_3 = h_2 h_3$ and $\Lambda = h_2^2 > 0$.

Finally, if t_0 was allowed to be zero, then $h_2 = h_3 = 0$ by (90), (73) and $x_1 x_2 x_3 \neq 0$. Taking the combination $x_3 \partial_1 q_1 - x_2 \partial_1 r_1$, using (103-104) and then (102), and finally inserting $q_1 = r_1 = 0$ (as follows from (94)) we arrive at $(n_1^2 - \Lambda)x_1 = 0$, contradictory to the fact that $\Lambda < 0$ when $h_2 = h_3 = 0$. \square

It follows from the proof that when $t_0 = 1$, the only allowed functional relations $\mathcal{F}(x_2, x_3) = 0$ are $x_2 = \pm\lambda^2$, $x_3 = \pm\lambda^2$ or (113), with λ a real constant. Thus, e.g., linear relations involving *both* x_2 and x_3 are inconsistent. In particular, all members of \mathcal{A} are of Petrov type $I(M^+)$ in the extended Arianrhod-McIntosh Petrov classification [28], i.e., the eigenvalues E_α are all non-zero.

For the non-expanding subclass of \mathcal{A} one derives from (77), (89) and (112) that

$$w = -8\Omega_1(\Omega_1 + \omega_1) = 2(\omega_1^2 + \sigma_{23}^2) \left(1 - \frac{\sigma_{23}^2}{\omega_1^2}\right). \quad (114)$$

This is positive if and only if $|\omega_1| > |\Omega_1|$ and $\Omega_1 \omega_1 < 0$, or if and only if

$$|\omega_1| > |\sigma_{23}| \Leftrightarrow \sqrt{\omega_{ab}\omega^{ab}} > \sqrt{\sigma_{ab}\sigma^{ab}}, \quad (115)$$

where $\omega_{ab} = u_{[a;b]} = \epsilon_{abc}\omega^c$ is the vorticity tensor. Thus *the energy density is positive if and only if the norm of the vorticity tensor is larger than the norm of the shear tensor*.

On the other hand, the expanding $t_0 = 1$ family is characterized by $|\omega_1| = |\sigma_{23}|$. When e.g. $V_1 = 0$ ($\sigma_{23} = \omega_1$) one may take $h_3 = -\lambda$ without loss of generality, which gives $x_2 = \lambda h_2$ and $x_3 = \lambda^2 > 0$. Still note from (90) that $x_3 = \text{const}$ is an equivalent characterization for this situation. We further deduce $\Omega_1 = 0$ from (88), $r_1 = q_3 = 0$ from (94-95), $n_1 = n_2 = n_3 = 0$ from (98) and (102), and finally $r_2 = 0$ by taking the ∂_2 -derivative of $x_2 = \lambda h_2$ and using (95), (99). Herewith the remaining equations are identically satisfied or determine consistent expressions for derivatives of the remaining variables h_2 , ω_1 and q_1 .

The invariant integration for this family was performed in [29]. The result is the line element

$$\begin{aligned} \lambda^2 ds^2 = & -(dt - 2y dz)^2 + (dx + f_1(z) dz)^2 \\ & + e^{2t} [dy + (f_2(z) + y^2 + e^{-2t}) dz]^2 \\ & + (\cos x - e^{-t})^2 f_3(z)^2 dz^2, \end{aligned} \quad (116)$$

where $\lambda = \sqrt{\Lambda}$ plays the role of a constant scaling factor and where t , x and y are invariantly defined coordinates. The arbitrary scalar functions $f_1(z)$, $f_2(z)$ and $f_3(z)$ are invariantly defined; only when these are all constant there is a continuous isometry group, which is one-dimensional and generated by $\partial/\partial z$. Writing $R \equiv \cos x e^t$ one has

$$\omega_1 = \frac{\lambda}{f_3(z)R}, \quad \mu = \frac{2\lambda^2}{R}, \quad \theta = \lambda \left(1 + \frac{1}{R}\right), \quad (117)$$

such that the matter density and expansion scalar are positive for $R > 0$.

5. LIMIT CASES

The special gravito-electric irrotational dust solutions found in [13], namely

$$\begin{aligned} \Lambda ds^2 = & -dt^2 + dx^2 + [e^{-t} + g_1(y)\cos x]^2 dy^2 \\ & + [e^t + g_2(z)\sin x]^2 dz^2, \end{aligned} \quad (118)$$

$$\begin{aligned} \Lambda ds^2 = & -dt^2 + dx^2 + [e^{-t} + e^{2t} dz^2 \\ & + g_1(y)\cos(x + g_2(y))]^2 dy^2, \end{aligned} \quad (119)$$

were precisely characterized by having a zero eigenvalue for the expansion tensor or, equivalently, by possessing a geodesic space-like Weyl principal vector field. In the above studied rotating case these properties have been shown to be satisfied automatically, the singled out WP vector field being moreover parallel to the vorticity. Thus, surprisingly, \mathcal{A} *consists exactly of the rotating generalizations of (118-119), which are on their turn contained as the irrotational limit solutions*. The metric

(116) is the rotating generalization of (119). As also follows from the analysis in [13], the relations (70-73), (75-78) and (79-81) with $\omega_1 = 0$ are valid for the metrics (118-119) when taking $\frac{\partial}{\partial t} \sim \partial_0$ and $\frac{\partial}{\partial x} \sim \partial_1$. Note from (80) that $\Lambda = \lambda^2 > 0$, where λ again plays the role of an overall scaling factor.

The shear-free limits $\sigma_{23} = h_2 = h_3 = 0$ of \mathcal{A} are non-expanding and automatically satisfy (115). We take (88) – instead of (79) – as a defining relation for \mathcal{A} , and derive from (??), (94), (101), (102), (88) and (77), respectively, that $x_1 = 0$, $n_2 = n_3 = q_1 = r_1 = 0$, $x_2 = \mu + \Lambda = \omega_1^2 = -\Lambda > 0$ and $2\Omega_1 = -\omega_1 = \lambda$. This corresponds precisely to the Gödel solution [30]

$$-2\Lambda ds^2 = dx^2 + dy^2 + \frac{1}{2}e^{2x}dz^2 - (dt + e^x dz)^2, \quad (120)$$

which may indeed be interpreted as dust $p = 0$ with cosmological constant Λ .

Nowhere in the reasoning leading to the results of the previous section we have explicitly used that $\mu \neq 0$. Because of (77) and the inconsistency of linear relations involving both x_2 and x_3 (cf. supra) it follows that Petrov type I Λ -terms cannot be contained in \mathcal{A} as ‘limits’ $\mu = 0$. Thus we have established:

Theorem 5.1 Purely electric, algebraically general Einstein spaces for which the time-like Weyl principal vector field is geodesic and rotating do not exist.

Mars [31] characterized the Kasner space-times [32, 33] as the purely electric Petrov type I vacua ($\mu = \Lambda = 0$) with non-rotating and geodesic time-like WP vector field. Combining the above theorem with this result and with further work on the non-rotating Einstein space case $\mu = 0 \neq \Lambda$ [46], the following conjecture may be stated:

Conjecture 5.2 The only purely electric, algebraically general Einstein spaces with a congruence of *freely falling* Weyl principal observers are the Kasner models and their generalizations including a cosmological constant.

6. CONCLUSION AND DISCUSSION

The following theorem summarizes the results obtained in this paper.

Theorem 6.1 Consider the class \mathcal{A} of algebraically general, gravito-electric rotating dust space-times. For any member of \mathcal{A} , the vorticity vector at each point is parallel to a Weyl principal vector which is moreover geodesic, the shear tensor does not commute with the Weyl electric tensor and is non-degenerate, the Petrov type is $I(M^+)$ in the extended Arianrhod-McIntosh Petrov classification, and the cosmological constant cannot vanish. The curvature variables E_{ab} , μ , Λ and kinematic quantities σ_{ab} , ω^a and θ are subject to the covariant algebraic relations

(82-86). The number t_0 of zero-order Riemann invariants is either 1 or 2. The subclass corresponding to $t_0 = 1$ splits into two separate parts. The first part consists of all non-expanding solutions ($\theta = 0$); these are Petrov type I , shearing generalizations of the Gödel universe, with $\Lambda < 0$, and the energy density is positive if and only if the norm of the vorticity tensor is strictly larger than the norm of the shear tensor. The second part corresponds precisely to the solutions for which these norms are equal, where now $\Lambda > 0$. The class does not allow for Einstein space (Λ -term) limits $\mu = 0$.

Note that these statements have been derived for dust ($p = 0$) space-times with cosmological constant, but they remain valid, *mutatis mutandis*, for perfect fluids with constant pressure. E.g., it has been proved that if an algebraically general, gravito-electric, non-expanding but rotating perfect fluid model has constant pressure p , then p must be larger than the cosmological constant present.

A second remark concerns the eventual relationship between the rotation of a congruence of observers u^a in a general space-time and the magnetic part of the Weyl tensor with respect to it [34]. Speculations about such a connection stem from the fact that the Bianchi ‘div \mathbf{H} ’ equation (for perfect fluids or Λ -terms: equation (21) with $\text{div } H_a$ instead of 0 in the right hand side), contains the ‘angular momentum density’ source term $(\mu + p)\omega_a$ – in contrast to the analogous Maxwell equation [35, 36]. The link was verified and affirmed, in some sense, for e.g. the van Stockum solution [37] [47] and the Bondi space-time [38]. However, examples are known of rotating gravito-electric perfect fluids [39, 40]. For such space-times the third term in (21) is exactly balanced by the first two terms. E.g., in the case of the Stephani-Barnes gravito-electric dust space-times of Petrov type D mentioned in the introduction, the rotation lies in the $E^a{}_b$ -eigenplane and one has $3E_{ab}\omega^b = -[\sigma, E]_a = -\mu\omega_a/2$. For the metrics of class \mathcal{A} considered in this paper, the rotation is an eigenvector of $E^a{}_b$ and there is an exact balancing based on (82), (83) and (85). In this respect, $\mu = -2\Lambda$ again leads to the homogeneous Gödel solution, where now the rotation vector of the dust is an $E^a{}_b$ -eigenvector corresponding the non-degenerate eigenvalue $2\Lambda/3$ and spans the axis of local rotational symmetry.

Another consequence of the analysis is the following restatement of the generalized silent universe conjecture (as put forward in [14]): *an algebraically general gravito-electric dust space-time is either an orthogonally spatially homogeneous Bianchi type I (OSH BI) Saunders model [12], or possesses a geodesic space-like $E^a{}_b$ -eigenvector field (and then belongs to \mathcal{A} or the non-rotating limit families (118-119)).* Conjecture 5.2 states that only OSH BI models are possible for the corresponding Λ -term limit.

By this investigation, the question by van Elst *et al.*, whether \mathcal{A} constitutes a broad class of solutions (cf. the introduction), may be answered affirmatively. We have seen that \mathcal{A} may be partitioned into three subclasses, one constituted by the non-expanding members, one by

the expanding members with $t_0 = 2$ and one by the expanding members with $t_0 = 1$. Equation (116) gives the general line element corresponding to the third subclass, which provides a first explicit example of algebraically general rotating dust with an at most one-dimensional isometry group, depending on three free functions of one coordinate.

On the other hand, the non-linearity of some of the class-defining algebraic relations hinders a transparent consistency analysis for the other two subclasses, at least in an orthonormal approach based on an E^a_b -eigentetrad. However, as somewhat hidden in such an approach, it turns out that there is a significant geometric duality between the dust four-velocity u^a and the normalized ro-

tation vector $v^a \equiv \omega^a / \sqrt{\omega_b \omega^b}$ at each space-time point: both are geodesic Weyl principal vectors, the vorticity vector of the one is parallel to the other [48], and the one is an eigenvector of the shear tensor of the other. These properties are most naturally expressed within a 1+1+2 covariant formalism, the first ‘1’ standing for u^a and the second ‘1’ for v^a , whereas ‘2’ expresses that one leaves a $SO(2, \mathbb{R})$ rotational freedom in the orthogonal complement of these vectors at each point. There is good hope that, on using a complexified version of such a formalism, one can elegantly tackle the general consistency problem for \mathcal{A} , and thereby substantiate that also the remaining subclasses $t_0 = 2$ and $t_0 = 1$, $\theta = 0$ contain a large number of metrics.

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- [1] Ellis G F R, Nel S D, Maartens R, Stoeger W R and Whitman A P *Phys. Rep.* **124**, 315 (1985)
 - [2] Maartens R and Matravers D R *Class. Quantum Grav.* **11**, 2693 (1994)
 - [3] Bertschinger E and Jain B *Astrophys. J.* **431**, 486 (1994)
 - [4] Ellis G F R and Dunsby P K S *Astrophys. J.* **479**, 97 (1997)
 - [5] Maartens R, Lesame W M and Ellis G F R *Class. Quantum Grav.* **15**, 1005 (1998)
 - [6] Matarrese S, Pantano O and Saez D *Phys. Rev. D* **47**, 1311 (1993)
 - [7] Bruni M, Matarrese S and Pantano O *Phys. Rev. Lett* **74**, 1916 (1995)
 - [8] Croudace K M, Parry J, Salopek D S, Stewart J M *AstroPhys J.* **423**, 22 (1994)
 - [9] Matarrese S, Pantano O and Saez D *Phys. Rev. Lett.* **72**, 320 (1994)
 - [10] van Elst H, Uggla C, Lesame W M, Ellis G F R and Maartens R *Class. Quantum Grav.* **14**, 1151 (1997)
 - [11] Sopuerta C F *Phys. Rev. D* **55**, 5936 (1997)
 - [12] Saunders P T *Non-isotropic model universes*, Ph.D. thesis, London (1967)
 - [13] Van den Bergh N and Wylleman L *Class. Quantum Grav.* **21**, 2291 (2004)
 - [14] Wylleman L *Class. Quantum Grav.* **23**, 2727 (2006)
 - [15] Apostolopoulos P S and Carot J, gr-qc/0605130
 - [16] Synge J L *Proc. London. Math. Soc.* **43**, 376 (1937)
 - [17] Winicour J *J. Math. Phys.* **16**, 1806 (1975)
 - [18] Stephani H, Kramer D, MacCallum M A H, Hoenselaers C and Herlt E, *Exact Solutions to Einstein's Field Equations (Second Edition)* (Cambridge: Cambridge University Press, 2003)
 - [19] Stephani H *Class. Quantum Grav.* **4**, 125 (1987)
 - [20] Barnes A *Class. Quantum Grav.* **16**, 919 (1999)
 - [21] Ehlers J *Akad. Wiss. Mainz Abh., Math.-Nat. Kl.* **11**, 793 (1961)
 - [22] Ellis G F R *General Relativity and Cosmology*, edited by R. K. Sachs (New York, Academic, 1971)
 - [23] Maartens R *Phys. Rev. D* **55**, 463 (1997)
 - [24] MacCallum M A H, *Cosmological Models from a Geometric Point of View (Cargèse)* Vol 6, page 61 (New York: Gordon and Breach, 1971)
 - [25] Cox D, Little J and O'Shea D, *Ideals, Varieties and algebras* (New York: Springer Verlag, 1992)
 - [26] van Elst H, *Extensions and applications of 1+3 decomposition methods in general relativity* (Ph.D. thesis, Queen Mary, London, 1996)
 - [27] van Elst H and Ellis G F R *Class. Quantum Grav.* **13**, 1099 (1996)
 - [28] Arianrhod R and McIntosh C B G *Class. Quantum Grav.* **9**, 1969 (1992)
 - [29] Wylleman L, *A Petrov type I and generically asymmetric rotating dust family*, submitted to *Class. Quantum Grav.* (gr-qc 0801.4766)
 - [30] Gödel H *Rev. Mod. Phys.* **21**, 447 (1949)
 - [31] Mars M *Class. Quantum Grav.* **16**, 3245 (1999)
 - [32] Kasner E *Amer. J. Math.* **43**, 217 (1921)
 - [33] Kasner E *Trans. A.M.S.* **27**, 155 (1925)
 - [34] Glass E N *J. Math. Phys.* **16**, 2361 (1975)
 - [35] Maartens R and Bassett B A *Class. Quantum Grav.* **15**, 705 (1998)
 - [36] Ellis G F R 1973 Cargèse Lectures in Physics, vol. VI, ed. E Schatzman (New York: Gordon and Breach)
 - [37] Bonnor W B *Class. Quantum Grav.* **12**, 1483 (1995)
 - [38] Herrera L, Santos N O and Carot J *J. Math. Phys.* **47**, 052502 (2006)
 - [39] Collins C B *J. Math. Phys.* **25**, 995 (1984)
 - [40] Sklavenites D *J. Math. Phys.* **26**, 2279 (1985)
 - [41] A recent attempt to prove this conjecture [15] contains conceptual mistakes, as will be commented on elsewhere.
 - [42] To the best of the author's knowledge.
 - [43] The Ω_α are the non-zero components of the angular velocity vector of the triad (∂_α) w.r.t. the ‘inertial compass’, see e.g. [24] and references therein.
 - [44] See e.g. [25], pp. 158.
 - [45] This commutator relation was given in [23], formula (A18), for the further subcase $\omega^a = 0$. In the general formula (B.14) of [26], for imperfect fluids and with H_{ab} and u^a not necessarily zero, the terms corresponding to $+3H_{c(a}S_{b)}^c$ have the wrong sign.
 - [46] In collaboration with N. Van den Bergh it has meanwhile been shown that there should be at least an Abelian G_2 isometry group. This will be presented elsewhere.
 - [47] The rotating dust interior of the van Stockum cylinder, when staying close to the axis, is purely electric w.r.t. a certain congruence of *non-comoving* observers.
 - [48] The vorticity of ∂_1^a may vanish.